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# Solvability of Commutative Right-Nilalgebras Satisfying $(b(aa))a = b((aa)a)$

## Abstract

We study commutative right-nilalgebras of right-nilindex four satisfying the identity  $(b(aa))a = b((aa)a)$ . Our main result is that these algebras are solvable and not necessarily nilpotent. Our results require characteristic  $\neq 2, 3, 5$ .

## Keywords

polynomial identity, nilpotency, solvability, right-nilagebra

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**SOLVABILITY OF COMMUTATIVE  
RIGHT-NILALGEBRAS SATISFYING  
 $(b(aa))a = b((aa)a)^*$**

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**Abstract**

*We study commutative right-nilalgebras of right-nilindex four satisfying the identity  $(b(aa))a = b((aa)a)$ . Our main result is that these algebras are solvable and not necessarily nilpotent. Our results require characteristic  $\neq 2, 3, 5$ .*

**2000 Mathematics Subject Classification :** *17 A30.*

**Key Words :** *Polynomial identity; Nilpotency; Solvability; Right-nilalgebra.*

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## 1. Introduction

Let  $A$  be a (possibly infinite dimensional) commutative right-nilalgebra of right-nilindex four over a field of characteristic  $\neq 2, 3, 5$  satisfying the identity

$$(b(aa))a = b((aa)a) \quad (1)$$

Right-nilindex four means that  $A$  satisfies the identity

$$((aa)a)a = 0. \quad (2)$$

Identity (1) appeared as an omitted special case in Hentzel and Labra[1]. In that paper they studied commutative algebras satisfying the polynomial identity

$$(b(aa))a + tb((aa)a) = 0$$

for  $t \neq 1, -1$ . It is immediate that every commutative algebra that satisfies this identity satisfies  $(t+1)((aa)a)a = 0$ . When  $t \neq -1$  the algebra automatically satisfies (2). When  $t = -1$  we get identity (1). We impose identity (2) because it is not a consequence of (1).

## 2. Preliminary Results

Linearizing completely (1) we get:

$$\begin{aligned} 0 = g(a, b, c, d) = & + (a(bc))d - a((bc)d) \\ & + (a(cd))b - a((cd)b) \\ & + (a(db))c - a((db)c). \end{aligned}$$

Linearizing completely (2) we get:

$$\begin{aligned} 0 = f(a, b, c, d) = & + ((ab)c)d + ((ab)d)c \\ & + ((ac)b)d + ((ac)d)b \\ & + ((ad)b)c + ((ad)c)b \\ & + ((bc)a)d + ((bc)d)a \\ & + ((bd)a)c + ((bd)c)a \\ & + ((cd)a)b + ((cd)b)a. \end{aligned}$$

**Proposition 1.** *Let  $A$  be a commutative algebra (possibly infinite dimensional) over a field of characteristic  $\neq 2, 3$ . Then  $A$  satisfies identities (1) and (2) if and only if  $A$  satisfies the identity*

$$((ba)a)a + b((aa)a) = 0. \quad (3)$$

**Proof.** We first assume that  $A$  satisfies the identities (1) and (2). From  $0 = f(b, a, a, a) = 6((ba)a)a + 3(b(aa))a + 3b((aa)a)$  we get:

$$2((ba)a)a + (b(aa))a + b((aa)a) = 0. \quad (4)$$

Using (1) and characteristic  $\neq 2$ , we get (3).

Now assume that  $A$  satisfies identity (3). We replace  $b$  by  $a$  in (3) and use characteristic  $\neq 2$  to obtain identity (2). Linearizing (2) we get (4). Subtracting two times identity (3) from identity (4) gives identity (1). This proves the Proposition.

### 3. Main Section

**Lemma 2.** *Let  $A$  be a commutative algebra (possibly infinite dimensional) over a field of characteristic  $\neq 2, 3$  satisfying identities (1) and (2). Then every subalgebra generated by a single element  $a$  of  $A$  is nilpotent of index at most 5. The only monomials that are not necessarily zero are:  $a, aa, (aa)a$  and  $(aa)(aa)$ .*

**Proof.** Let  $a$  be an element of  $A$ . We will prove that all the (non associative) monomials that have degree  $\geq 5$  are zero. The proof proceeds by induction using degree 5 as the start.

The degree 5 monomials are:  $m_1 = (((aa)a)a)a$ ,  $m_2 = ((aa)(aa))a$ , and  $m_3 = ((aa)a)(aa)$ .

Monomial  $m_1$  is zero from (2). Using (1), (3) and (2):  $m_2 = ((aa)(aa))a = (aa)((aa)a) = -(((aa)a)a)a = 0$ . Using commutativity, (3), and (2):  $m_3 = ((aa)a)(aa) = (aa)((aa)a) = -(((aa)a)a)a = 0$ .

The degree 6 monomials which are not immediately zero by (2) and induction are:  $m_4 = ((aa)(aa))(aa)$  and  $m_5 = ((aa)a)((aa)a)$ .

Using (3) and (2):  $m_5 = ((aa)a)((aa)a) = -((((aa)a)a)a)a = 0$ . The identity  $f(a, a, aa, aa)$  establishes a dependence relation between  $m_4 = ((aa)(aa))(aa)$  and other monomials of degree 6. Since these other degree 6 monomials are zero,  $m_4$  is zero as well.

The only degree 7 monomial which is not zero by induction and (2) is  $m_6 = ((aa)(aa))((aa)a)$ .

Using (3) and induction:  $m_6 = ((aa)(aa))((aa)a) = -((((aa)(aa))a)a)a = 0$ .

The only degree 8 monomial which is not zero by induction and (2) is  $m_7 = ((aa)(aa))((aa)(aa))$ .

The identity  $f(a, a, aa, (aa)(aa))$  establishes a dependence relation between  $m_7 = ((aa)(aa))((aa)(aa))$  and other monomials of degree 8. Since all the other monomials are zero,  $m_7$  is zero as well.

Monomials of degree higher than 8 are zero by induction because one of the factors will have degree greater than 4.

**Lemma 3.** If  $A$  is commutative algebra that satisfies (1) and (2), then  $48((aa)a)((bb)b) = 0$  for all  $a, b \in A$ .

**Proof.** Let us consider the following monomials:

$$\begin{aligned}
b_1 &= (((aa)a)b)b & b_{14} &= (((ab)a)b)a \\
b_2 &= (((aa)(ab))b)b & b_{15} &= (((ab)a)b)b \\
b_3 &= (((aa)a)(bb))b & b_{16} &= (((ab)b)a)a \\
b_4 &= (((aa)b)a)b & b_{17} &= (((ab)b)(aa))b \\
b_5 &= (((aa)b)(ab))b & b_{18} &= (((ab)b)a)b \\
b_6 &= ((aa)b)((ab)b) & b_{19} &= (((bb)a)a)a \\
b_7 &= (((aa)b)b)a & b_{20} &= (((bb)a)(aa))b \\
b_8 &= (((aa)(bb))a)b & b_{21} &= (((bb)a)a)b \\
b_9 &= (((aa)b)b)a & b_{22} &= (((bb)a)b)a \\
b_{10} &= (((aa)(bb))b)a & b_{23} &= (((bb)b)a)a \\
b_{11} &= (((aa)b)(bb))a & b_{24} &= (((bb)b)(aa))a \\
b_{12} &= ((aa)b)((bb)a) & b_{25} &= ((aa)a)((bb)b) \\
b_{13} &= (((ab)a)a)b
\end{aligned}$$

Straightforward calculations show that the following linear combination of identities  $f$  and  $g$  produces the result that  $48((aa)a)((bb)b) = 0$  :

$$\begin{aligned}
+27f((aa)a, b, b, b) &= 162b_1 + 81b_3 + 81b_{25} \\
+30f((aa)b, a, b, b) &= 60b_4 + 60b_5 + 60b_6 + 60b_7 + 60b_9 + 30b_{11} + 30b_{12} \\
+7f((bb)b, a, a, a) &= 42b_{23} + 21b_{24} + 21b_{25} \\
-30f(aa, a, b, b)b &= -60b_1 - 60b_2 - 60b_4 - 60b_7 - 30b_8 - 60b_{17} - 30b_{20} \\
-15f(aa, b, b, b)a &= -90b_9 - 45b_{10} - 45b_{24} \\
-11f(bb, a, a, a)b &= -33b_3 - 33b_8 - 66b_{19} \\
-16(f(a, a, a, b)b)b &= -48b_1 - 48b_4 - 96b_{13} \\
-6(f(a, a, b, b)a)b &= -12b_7 - 24b_{14} - 24b_{16} - 12b_{19} \\
+18(f(a, a, b, b)b)a &= 36b_9 + 72b_{15} + 72b_{18} + 36b_{21} \\
+2g((aa)a, b, b, b) &= 6b_3 - 6b_{25}
\end{aligned}$$

$$\begin{aligned}
-9g((aa)b, a, b, b) &= -18b_5 + 18b_6 - 9b_{11} + 9b_{12} \\
+16g((bb)b, a, a, a) &= 48b_{24} - 48b_{25} \\
-6g(b, (aa)b, a, b) &= 6b_5 - 6b_6 + 6b_7 - 6b_9 \\
-72g(b, (ab)b, a, a) &= -72b_6 + 144b_{16} + 72b_{17} - 144b_{18} \\
-39g(b, (bb)a, a, a) &= -39b_{12} + 78b_{19} + 39b_{20} - 78b_{21} \\
+6g(aa, a, b, b)b &= 12b_2 + 6b_8 - 12b_{17} - 6b_{20} \\
-3g(a, aa, b, b)b &= -6b_4 + 6b_7 + 3b_8 - 3b_{20} \\
-54g(a, a, a, bb)b &= -54b_3 + 54b_8 \\
-24g(b, aa, b, b)a &= 24b_{10} - 24b_{24} \\
-21g(b, a, a, bb)a &= 21b_{10} - 21b_{11} + 42b_{21} - 42b_{22} \\
-48g(b, ab, a, a)b &= 48b_2 - 48b_5 + 96b_{13} - 96b_{14} \\
+14g(a, b, b, b)a &= 42b_{22} - 42b_{23} \\
+18g(b, a, a, a)b &= -54b_1 + 54b_4 \\
-60g(b, a, a, b)a &= 120b_{14} - 120b_{16} \\
+36g(b, a, a, b)b &= -72b_{15} + 72b_{18}
\end{aligned}$$

This proves the Lemma.

**Theorem 4.** *Let  $A$  be a commutative algebra over a field of characteristic  $\neq 2, 3$  that satisfies identities (1) and (2). Let  $W$  be the linear subspace of  $A$  generated by the elements of the form  $J(a, b, c)$ , where  $J(a, b, c) = (ab)c + (bc)a + (ca)b$ , with  $a, b, c$  in  $A$ . Then  $W$  is an ideal of  $A$  with  $W^2 = 0$ .*

**Proof.** From Lemma 3 and characteristic  $\neq 2, 3$ , we get that  $((aa)a)((bb)b)$  is an identity in  $A$ . Linearizing this identity we get that  $J(a, b, c) \cdot J(d, e, f) = 0$  for any elements  $a, b, c, d, e, f$  in  $A$ . This proves that  $W^2 = 0$ .

On the other hand, we see that for every  $a$  and  $b$  in  $A$ , the product  $b((aa)a)$  is in  $J(A, A, A) \subset W$ . This proof follows below.

$$\begin{aligned}
&J(b, aa, a) - J(ba, a, a) \\
&= (b(aa))a + ((aa)a)b + (ab)(aa) - ((ba)a)a - (aa)(ba) - (a(ba))a \\
&= (b(aa))a + b((aa)a) + (ba)(aa) - ((ba)a)a - (ba)(aa) - ((ba)a)a \\
&= 4b((aa)a) \text{ using (1) and (3).}
\end{aligned}$$

Hence,  $b((aa)a)$  is in  $W$  for every  $a$  and  $b$  in  $A$ , which proves Theorem 4.

**Theorem 5.** *Let  $A$  be a (possibly infinite dimensional) commutative algebra over a field of characteristic  $\neq 2, 3, 5$  satisfying identities (1) and (2). Then  $A$  is solvable.*

**Proof.** Using Theorem 5 we get that the quotient  $A/W$  is an algebra that satisfies the polynomial identity  $J(a, b, c) = 0$ . In particular,  $A/W$

satisfies the identity  $a^3 = 0$ . It is known that every commutative algebra that satisfies this identity is a Jordan algebra. Hence,  $A/W$  is a Jordan nilalgebra of nilindex three. From Zelmanov and Skosyrskii[5], we get that  $A/W$  is solvable. Since  $W^2 = 0$ , we have that  $A/W$  and  $W$  are solvable. Therefore, from Schafer[3] (Proposition 2.2, p.18), we get that  $A$  is solvable, which proves Theorem 5. The proof of Zelmanov and Skosyrskii[5] uses characteristic  $\neq 2, 3, 5$ .

We remark that algebras with the hypotheses of Theorem 5 are not necessarily nilpotent. In fact, the algebra  $A$  constructed by Zhevlakov in [4](Example 1, p.82), is a commutative algebra that satisfies the polynomial identity  $a^3 = 0$  and is solvable with  $A^2A^2 = 0$  but is not nilpotent. It is easy to prove that  $A$  satisfies identity (1).

We wish to remark that the computer algebra system ALBERT[2] indicates that any commutative algebra  $A$  satisfying the identity  $(xx)x = 0$  will also satisfy:  $((AA)(AA))((AA)(AA))A = 0$ . This indicates that  $((AA)(AA))((AA)(AA))$  annihilates the whole algebra and that  $A$  is solvable of index  $\leq 5$ .

This implies that a commutative algebra satisfying identities (1) and (2) will be solvable of index  $\leq 6$ . ALBERT shows that for these algebras  $((AA)(AA))((AA)(AA))A$  need not be zero. This shows that they are not all solvable of index  $\leq 4$ . This does not rule out the possibility that they might all be solvable of index  $\leq 5$ . This work with ALBERT indicates that the assumption of characteristic  $\neq 5$  may be unnecessary.

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